Latitude dependence of the maximum duration of a total solar eclipse

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Introduction

It is well known that the maximum duration of a total eclipse (in the current epoch) is just over 7m 30s (450s), in Morsels III Jean Meeus gives the value as 7m 32s (452s). It is obvious that the longest duration will occur when the Earth is near aphelion (currently around 5 July), and that it will therefore happen slightly north of equator. A latitude of 5° north was mentioned by Danjon on page 318 in "Astronomie Générale" (Paris, 1952). However, the more general question: "What is the maximum duration of a total solar eclipse as a function of the latitude?" does not appear to have been addressed. As a detailed answer to this question would be an arduous task, we are instead using simpler methods to derive a reasonably accurate estimate. The first approach is to look at the data from the Five Millennium Canon. The second is to devise a highly simplified mathematical model to provide an approximate result. Finally, the third approach is to look at the duration of "artificial" eclipses; these are based on "real" eclipses, but their central lines have been displaced.

Data from the Five Millennium Canon

Figure 1 shows duration versus latitude for all the total eclipses in the Five Millennium Canon. We notice in particular the maximum duration of 449s. This happens on 17th July 2186 at a latitude of 7.4° north. This duration is very close to the theoretical maximum of 452s.

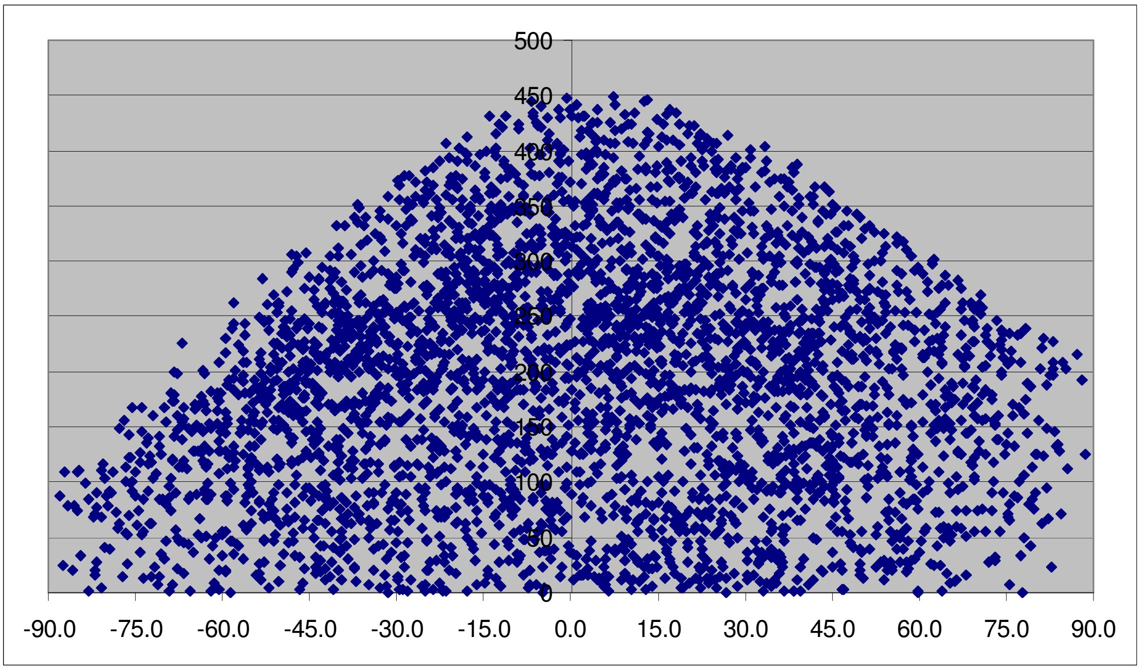

Figure 1. Scatter plot of eclipse duration in seconds vs latitude in degrees (-90 corresponds to the South Pole, +90 to the North Pole). Data from the Five Millennium Canon.

Although figure 1 qualitatively agrees with the expectations, and quantitatively with the absolute maximum duration, we cannot be sure that this finite data set captures the maximum duration for all values of the latitude. This qualification is particularly important for the polar regions where there are far fewer data points.

Simple mathematical model

The first step is to find an expression for the eclipse magnitude. We start by writing the angular size of the Sun as

$$\Theta_s = \Theta_{s_0} - \delta\Theta \cos(\varphi - \varphi_s) \qquad (1)$$

In this expression Θ_{s_0} is the average angular size, $\delta\Theta$ is the amplitude of the variation in the angular size, and φ describes the position of the Earth in its orbit (i.e. the date) with $\varphi = \varphi_s$ when the Earth is at aphelion.

The angular size of the Moon at perigee, *as seen from the* *centre* *of the Earth* is given by

$$\Theta_m = \frac{d_{Moon}}{d_m} \qquad (2)$$

where d_{Moon} is the diameter of the Moon and d_m is the minimum distance between the centre of the Earth and the centre of the moon at perigee.

We can now write the eclipse magnitude with the Sun at its average distance and the Moon at perigee as

$$M_0 = \frac{\Theta_m}{\Theta_{s_0}} \qquad (3)$$

with a date dependent correction for the variation of the angular size of the Sun given by

$$M_1 = \frac{\delta\Theta}{\Theta_{s_0}} \cos(\varphi - \varphi_s) \qquad (4)$$

Next we express the declination of the axis of the Earth with respect to the Sun as

$$d = d_{max} \cos(\varphi - \varphi_d) \qquad (5)$$

Here d_{max} is the maximum declination (23.5°), and again φ describes the position of the Earth with $\varphi = \varphi_d$ at Summer solstice

The minimum distance D from a point at latitude Θ *on the* *surface* *of the Earth* to the centre of the Moon, with the Moon at perigee, a declination d, and the Earth radius r_E is given by

$$D = \sqrt{d_m{}^2 - 2r_E d_m \cos(\Theta - d) + r_E{}^2} \qquad (6)$$

which can be approximated as

$$D \approx d_m \ \left(1 - \frac{r_E}{d_m}\cos(\Theta - d)\right) \tag{7}$$

as D differs from d_m used in (2), we get an additional magnitude correction given by

$$M_2 = \frac{r_E}{d_m}\cos(\Theta - d) \tag{8}$$

We finally find the magnitude as a function of the latitude (Θ) and the date (through φ) by combining the above expressions

$$M(\Theta, \varphi) = M_0 + M_1 + M_2 \tag{9}$$

The maximum magnitude $M_{max}(\Theta)$ at a given latitude Θ can now be found by maximising M with respect to φ. Note that in this operation we need to exclude values of φ which correspond to the Sun being below the horizon. On the southern hemisphere this happens for $d > 90^0 + \Theta$.

The following values have been used for the various constants: $M_0 = \frac{\Theta_m}{\Theta_{s_0}} = 1.04835$, $\frac{\delta\Theta}{\Theta_{s_0}} = 0.0179$, $\frac{r_E}{d_m} = 0.0167$, $\varphi_s = 3.20$ rad (corresponding to 5 July) and $\varphi_d = 2.96$ rad (corresponding to 21 June).

The second step is to translate the eclipse magnitude into an eclipse duration, using the maximum magnitude for each latitude. First we use a simple estimate of the average angular velocity of the Moon (seen from the Earth) relative to that of the Sun

$$v_a' = \frac{360^0}{29.5\text{d} \ \ 24\text{h/d} \ \ 3600\text{s/h}} = 1.412 \ \ 10^{-4} \ {}^0/\text{s} \tag{10}$$

However, we need to correct for the fact that for a maximum duration eclipse the Earth is near aphelion and the Moon is near perigee. This means that the angular velocity of the Sun is lower, and the angular velocity of the Moon is higher. According to Keppler’s second law, velocity times distance is constant, but since *angular* velocity equals velocity divided by distance it follows that angular velocity equals a constant divided by the *square* of the distance. The combined effect of a higher distance to the Sun (and hence a lower angular velocity of the Sun), and a lower distance to the Moon (and hence a higher angular velocity of the Moon), and the dependence of the square of the distance means that the relevant relative angular velocity of the Moon is higher than the value given by equation (10). A more accurate value is given by

$$v_a = v_a' 1.18 \tag{11}$$

The final step is to include the influence of the rotation of the Earth. The duration of an eclipse for a static Earth (or at the poles) is

$$Du'(\Theta) = (M_{max}(\Theta) - 1)\frac{\Theta_s'}{v_a} = (M_{max}(\Theta) - 1)\, 3144\ \mathrm{s} \qquad (12)$$

Where $\Theta_s' = 0.524°$ is the minimum value of the angular size of the Sun.

With v_E being the surface velocity of the Earth at equator (40,000km/24h) and v_M being the velocity of the Moon, we finally get the eclipse duration at the latitude Θ

$$Du(\Theta) = Du'(\Theta)(1 - \frac{v_E}{v_M}\cos(\Theta))^{-1} = Du'(\Theta)(1 - 0.4287\cos(\Theta))^{-1} \qquad (13)$$

Numerical implementation and results

The required numerical calculations are all straight forward and conveniently carried out in a simple spreadsheet. The first step is to generate a 2-D table of the magnitude M as a function of the date and the latitude based on equations (1)-(9). For each value of the latitude the maximum magnitude can then be found by interpolation.

As mentioned above, we need to exclude periods where the Sun is below the horizon. For the northern hemisphere this is not an issue since the maximum magnitude happens in July. However, close to the South Pole the Sun is below the horizon at the time of maximum magnitude, so the maximum "visible" magnitude will occur when the Sun is just on the horizon at mid-day. The most extreme case is the South Pole itself, where the maximum visible magnitude will occur at the autumnal equinox.

Using equations (10)-(13) we find the maximum duration as a function of the latitude shown in figure 2.

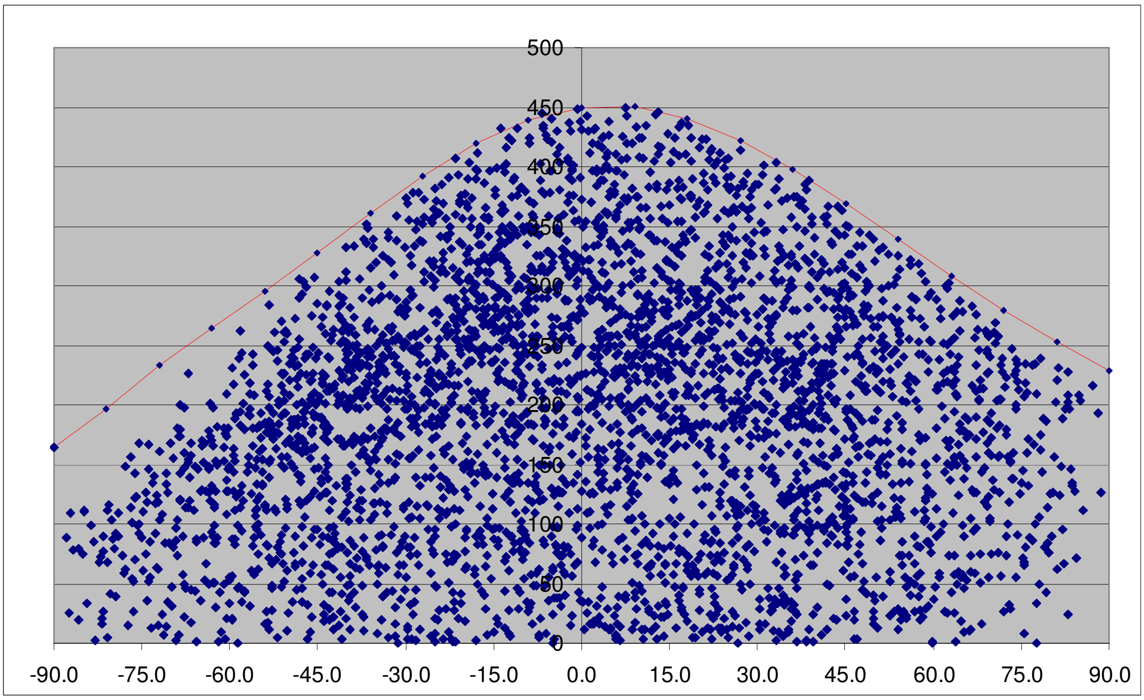

Figure 2. Calculated maximum total eclipse duration as a function of latitude (red curve with small blue markers) superimposed on the scatter plot from figure 1. There is a very slight kink near -70° because we have to use the highest value for the *visible* magnitude.

It is seen that the calculated results agree extremely well with the Five Millennium data near equator, and that there is a reasonable agreement for most of the northern hemisphere. However, in the polar regions, in particular near the South Pole, the agreement is less good.

At this point of the investigation it is not obvious whether this apparent disagreement for latitudes near the South Pole is caused by deficiencies in the model, or whether it is because eclipses with a duration close to the maximum at these latitudes are so rare that they do not occur in the period covered by the Five Millennium Canon.

Artificial eclipses, circumstances at the South Pole

In order to clarify the situation near the South Pole we consider eclipses satisfying the following conditions:
(1) the Sun is above the horizon,
(2) the eclipse is total in the fundamental plane
(3) Gamma is between -0.6 and -1.2.
For each of these eclipses the central line is "shifted" so it passes over the South Pole and the duration is calculated.

The results are summarised in the following table.

Period	Year and date of eclipse	Maximum duration
Years 1 to 1000	405, 16 March	157s
1001 - 2000	1913, 30 September	148s
2001 - 3000	2285, 29 September	158s

We see that the last two cases occur just after the autumnal equinox, whereas the first case occurs just before the vernal equinox. The reason for this is that in the year 1246 aphelion coincided with the Summer solstice. This corresponds to the value of φ_s being equal to the value of φ_d. Before 1246 aphelion was closer to the vernal equinox, after 1246 it has been closer to the autumnal equinox. The effect of this can easily be studied with the simple model, the results are shown in figure 3.

The results from the study of the artificial eclipses are reasonably close to those predicted by the simple model, in particular if we take into account that the duration of the artificial eclipses could be longer by up to 5s had they occurred even closer to the equinox. This indicates that the predicted 164s maximum duration (in the current epoch) for a total solar eclipse at the South Pole predicted by the simple model, is a reasonable estimate.

For the North Pole the situation is much simpler since the maximum magnitude and duration will happen near Summer solstice. A similar study of artificial eclipses, but with Gamma now restricted to the interval +1.1 to +0.6, gave a maximum duration of 218s in the period 1001 to 2000, and 216s for the period 2001 to 3000. This is in quite good agreement with the result of 229s from the simple model (see figure 2).

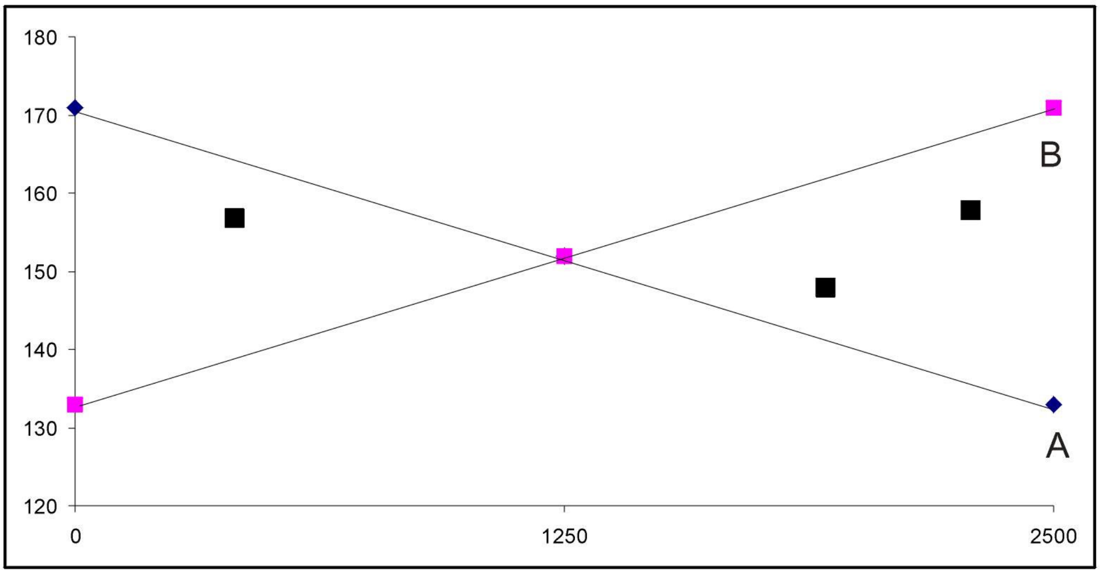

Figure 3. Maximum duration in seconds of a total solar eclipse at the South Pole as function of the year. A) at vernal equinox. B) at autumnal equinox. The three black square points indicate the results from the artificial eclipses.

Conclusion

The findings can be summarised as follows: We have developed a simple model for the calculation of the maximum duration of a total solar eclipse as a function of the latitude. The results from this model are in reasonable agreement with data from the Five Millennium Canon, except in the polar regions. However, a study based on artificial eclipses indicate that the apparent discrepancies (in particular in the southern hemisphere) are simply due to the fact that at these latitudes eclipses with a duration close to the maximum are very rare, and have not occurred in the period covered by the Canon.

We note that the circumstances near the South Pole are significantly more complicated than at other latitudes, with the maximum possible duration varying significantly over time (about 1.5s per century). For other latitudes the eclipse magnitude as a function of the date has a fairly flat maximum; consequently there is little change in the maximum duration when the separation between aphelion and Summer solstice changes.

All in all, the simple model seems to be correct to within a few seconds, and we have thus established the desired approximate results for the duration of a total eclipse as a function of the latitude.

Acknowledgement

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